

# CENTRAL POINTS AND MEASURES AND DENSE SUBSETS OF COMPACT METRIC SPACES

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**ABSTRACT.** For every nonempty compact convex subset  $K$  of a normed linear space a (unique) point  $c_K \in K$ , called the generalized Chebyshev center, is distinguished. It is shown that  $c_K$  is a common fixed point for the isometry group of the metric space  $K$ . With use of the generalized Chebyshev centers, the central measure  $\mu_X$  of an arbitrary compact metric space  $X$  is defined. For a large class of compact metric spaces, including the interval  $[0, 1]$  and all compact metric groups, another ‘central’ measure is distinguished, which turns out to coincide with the Lebesgue measure and the Haar one for the interval and a compact metric group, respectively. An idea of distinguishing infinitely many points forming a dense subset of an arbitrary compact metric space is also presented.

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**Key words:** Chebyshev center, convex set, common fixed point, Kantorovich metric, pointed metric space, distinguishing a point.

## 1. INTRODUCTION

Distinguishing points, subsets or other ‘ingredients’ related to spaces is important in many parts of mathematics, including algebraic topology (homotopy groups), theory of Lipschitz functions (the base point), theory of locally compact groups (the Haar measure, unique up to a constant factor). In most of algebraic structures the neutral element is a naturally distinguished point. In other areas of mathematics distinguishing appears as a useful tool. For example, the well-known *Chebyshev center* of a nonempty compact convex subset of a **strictly** convex normed linear space (i.e. such a space in which the unit sphere contains no segments [6, page 30]) finds an application in fixed point theory, being a common fixed point for the isometry group of the convex set. The characteristic and important feature of some of the above examples is the uniqueness, in a categorical or weaker sense, of the distinguished ingredients. In such cases this distinguished ingredient may be seen as an integral part of the space (e.g. the Haar measure of a locally compact group or the neutral element of an algebraic structure), while in the others it plays an additional role (e.g. in homotopy groups, spaces of Lipschitz functions). In the latter cases the distinguishing is just a necessity and it hardly ever finds applications. The foregoing examples show that the situation changes when the distinguished ingredient

turns out to be uniquely determined by some natural conditions. (The precise meaning of this in category of metric spaces shall be explained in the next section.)

The aim of the recent paper is to present a few results dealing with constructive ‘applied’ distinguishings. In particular, we shall show that every nonempty compact metric space  $X$  is isometric to a convex subset of a normed linear space (even of a more general class, containing all metric  $\mathbb{R}$ -trees) contains a unique point  $c_X$  (called the *generalized Chebyshev center*) which is in a sense its center. As an application of this, we shall prove that the isometry group of each such space has a common fixed point. This gives a constructive proof of Kakutani’s fixed point theorem in a special case. Details are included in Section 3.

In Section 4 we shall apply the results of the previous part to an arbitrary (nonempty) compact metric space  $X$  in order to define the *central* (probability Borel) measure  $\mu_X$  of  $X$  by means of the so-called Kantorovich (or Kantorovich-Rubenstein, cf. [21]) metric induced by the metric of  $X$ . In case of a compact metric group  $G$ ,  $\mu_G$  turns out to be the Haar measure of  $G$  and thus we shall obtain an alternative proof of the Haar measure theorem for compact metrizable groups. However, the problem of whether  $\mu_{[0,1]}$  is the one-dimensional Lebesgue measure we leave as open. Section 5 deals with the so-called *quasi-nilpotent* compact metric spaces for which we shall prove another result on distinguishing measures. As a special case we shall obtain the characterizations of the Lebesgue measure on  $[0, 1]$  and (again) the Haar measure of a compact metric group. The last, sixth, part is devoted to distinguishing countable dense subsets in arbitrary compact metric spaces, which is related to theory of random metric spaces (see e.g. [19, 20]).

## 2. PRELIMINARIES

In this paper we deal with categories of metric spaces with additional structures in which every *isomorphism* between spaces is an isometric function between them. For simplicity, let us call each such a category an *iso-category*. We shall write  $K \in \mathcal{K}$  to express that  $K$  is a metric space with an additional structure which belongs to an iso-category  $\mathcal{K}$ .

Let  $\mathcal{K}$  be an iso-category. For any two members  $X$  and  $Y$  of  $\mathcal{K}$  let  $\text{Iso}_{\mathcal{K}}(X, Y)$  stand for the set of all isomorphisms of  $X$  onto  $Y$ . We write  $\text{Iso}_{\mathcal{K}}(X)$  for  $\text{Iso}_{\mathcal{K}}(X, X)$ . If no additional structures on metric spaces are needed to describe the category  $\mathcal{K}$ , we shall write simply  $\text{Iso}(X, Y)$  and  $\text{Iso}(X)$ .

For  $X \in \mathcal{K}$  let ‘ $\sim_{\mathcal{K}}$ ’ be the equivalence relation on  $X$  given by

$$x \sim_{\mathcal{K}} y \iff \Phi(x) = y \text{ for some } \Phi \in \text{Iso}_{\mathcal{K}}(X);$$

let  $X^{(1)}$  be the quotient set  $X / \sim_{\mathcal{K}}$  and  $\pi_X^{(1)}: X \rightarrow X^{(1)}$  the canonical projection. Similarly, for any isomorphism  $\Phi \in \text{Iso}_{\mathcal{K}}(Y, Z)$  between

spaces  $Y, Z \in \mathcal{K}$  let  $\Phi^{(1)}: Y^{(1)} \rightarrow Z^{(1)}$  be the unique function such that  $\pi_Z^{(1)} \circ \Phi = \Phi^{(1)} \circ \pi_Y^{(1)}$ .

**2.1. Definition.** Let  $\mathcal{K}$  be an iso-category. By a (*weak*) *distinguishing* in  $\mathcal{K}$  we mean any assignment  $\mathcal{K} \ni X \mapsto C_X \in X^{(1)}$  such that whenever  $\Phi \in \text{Iso}_{\mathcal{K}}(K, L)$  with  $K, L \in \mathcal{K}$ , then  $\Phi^{(1)}(C_K) = C_L$ .

More natural approach to distinguishing is the following: to each space  $X \in \mathcal{K}$  assign a point  $c_X \in X$  in such a way that whenever  $K$  and  $L$  are two isomorphic members of  $\mathcal{K}$ , there is an isomorphism  $\Phi: K \rightarrow L$  which sends  $c_K$  to  $c_L$ . However, we are interested in constructive methods of distinguishing (so, without using the axiom of choice) and thus the original definition (Definition 2.1) is more appropriate.

A very special and the most important case of distinguishing appears when the distinguished equivalence class  $C_K$  consists of a single point and then we may consider  $C_K$  as an element of  $K$ . To make this precise, we put

**2.2. Definition.** By a *strict distinguishing* in an iso-category  $\mathcal{K}$  we mean any assignment  $\mathcal{K} \ni X \mapsto c_X \in X$  such that  $\Phi(c_K) = c_L$  for any  $K, L \in \mathcal{K}$  and each  $\Phi \in \text{Iso}_{\mathcal{K}}(K, L)$ .

Strict distinguishings appear very rarely in mathematics, which the following immediate result witnesses to

**2.3. Proposition.** *If  $\mathcal{K} \ni X \mapsto c_X \in X$  is a strict distinguishing in an iso-category  $\mathcal{K}$ , then for every  $K \in \mathcal{K}$ ,  $c_K$  is a common fixed point for the group  $\text{Iso}_{\mathcal{K}}(K)$ . That is,  $\Phi(c_K) = c_K$  for all  $\Phi \in \text{Iso}_{\mathcal{K}}(K)$ .*

Since there are iso-categories  $\mathcal{K}$  (even among those of nonempty compact spaces) in which for some spaces  $K \in \mathcal{K}$  the group  $\text{Iso}_{\mathcal{K}}(K)$  has no common fixed point, a strict distinguishing is not always possible. In the next section we introduce an iso-category for which the latter is realizable.

### 3. WEAKLY CONVEX COMPACT METRIC SPACES

In the literature there are two main approaches to the notion of convexity in metric spaces. The first is related to joining points by line segments, the second relies on generalization of the notion of the middle point between two points by describing its global position in the space. For example, Takahashi [18] calls a metric space  $(X, d)$  convex iff for any  $x, y \in X$  and every  $\lambda \in (0, 1)$  there is a point  $z_\lambda \in X$  such that

$$(3-1) \quad d(z_\lambda, w) \leq (1 - \lambda)d(x, w) + \lambda d(y, w)$$

for all  $w \in X$ ; Kijima [8] and Yang and Zhang [22] speak about convexity when (3-1) with  $\lambda = \frac{1}{2}$  is fulfilled; while Kindler [9] says about  $\varphi$ -convexity for any concave, nondecreasing in both variables function

$\varphi$  such that  $\varphi(x, y) < \max(x, y)$  whenever  $x \neq y$ . The reader interested in this topic is referred to the original papers of the above mentioned authors. Below we introduce the so-called weakly convex metric spaces, the class of which includes all known to us convex metric spaces defined by generalizing the notion of the middle point.

**3.1. Definition.** A metric space  $(X, d)$  is said to be *weakly convex* iff for any two points  $x$  and  $y$  of  $X$  there is a point  $z \in X$  such that for each  $w \in X$ :

$$(C1) \quad d(z, w) \leq \max(d(x, w), d(y, w)),$$

$$(C2) \quad d(x, w) = d(y, w) \text{ provided } d(z, w) = \max(d(x, w), d(y, w)).$$

Every point  $z \in X$  which satisfies (C1) and (C2) for fixed  $x, y \in X$  and all  $w \in X$  is said to be a *weakly middle point between  $x$  and  $y$* .

The reader will easily check that if  $X$  is a convex subset of a normed linear space, then  $X$  is weakly convex in the sense of Definition 3.1 (for  $x$  and  $y \in X$  the point  $z = \frac{x+y}{2}$  satisfies the conditions (C1) and (C2)). It may also be shown that every  $\mathbb{R}$ -tree is weakly convex. (A complete metric space  $T$  is said to be an  $\mathbb{R}$ -tree if for any two distinct points  $x$  and  $y$  of  $T$  there is a unique homeomorphic copy  $\gamma_{x,y}$  of the interval  $[0, 1]$  which joins  $x$  and  $y$ , and  $\gamma_{x,y}$  is isometric to a line segment; cf. [10].)

Our aim is to construct a strict distinguishing in the class  $\mathcal{WCC}$  of all nonempty weakly convex compact metric spaces (where the category is determined only by metrics). As a corollary, we shall obtain a theorem on common fixed points in weakly convex compact metric spaces.

Let  $(X, d)$  be a weakly convex metric space. For each  $x, y \in X$  let  $M(x, y)$  be the set of all weakly middle points between  $x$  and  $y$  in  $X$ . A subset  $A$  of  $X$  is said to be a *fully convex subspace* of  $X$  iff  $M(a, b) \subset A$  for all  $a, b \in A$ . It is clear that a fully convex subspace of a weakly convex metric space is itself a weakly convex metric space as well.

Now we shall recall the classical attributes of a metric space (see e.g. [3] or [9]). By  $\delta(X)$  we denote the *diameter* of a metric space  $(X, d)$ , that is,  $\delta(X) := \sup_{x,y \in X} d(x, y) \in [0, +\infty]$  provided  $X$  is nonempty and  $\delta(\emptyset) := 0$ . For each  $x \in X$  let  $r_X(x) := \sup_{y \in X} d(x, y)$  and let  $r(X) := \inf_{x \in X} r_X(x)$  ( $r(\emptyset) := 0$ ). The number  $r(X) \in [0, +\infty]$  is called the *Chebyshev radius* of  $X$ . Finally, the *Chebyshev center* of  $X$  is the set  $C(X) := \{x \in X : r_X(x) = r(X)\}$ . If the latter set consists of a single point, the unique element of  $C(X)$  is also called the *Chebyshev center* of  $X$ . The classical result states that  $C(K)$  is a singleton provided  $K$  is a nonempty compact convex subset of a strictly convex normed linear space. If the assumption of strict convexity of the norm is relaxed, the set  $C(K)$  may be infinite. However,  $C(X)$  is nonempty for every nonempty compact metric space  $X$ .

In order to define the generalized Chebyshev center (as a uniquely determined point of a space), we introduce the following

**3.2. Definition.** The  $n$ -th Chebyshev center,  $C^n(X)$ , of a metric space  $X$  is given by the recursive formula:  $C^0(X) := X$  and  $C^n(X) := C(C^{n-1}(X))$  for  $n > 0$ . Additionally, let  $C^\infty(X) := \bigcap_{n=0}^\infty C^n(X)$ .

Our goal is to show that  $C^\infty(X)$  consists of a single point provided  $X$  is a nonempty weakly convex compact metric space. To show this, we need the next result. It was proved (in a different way) in special cases by Takahashi [18] and Kindler [9].

**3.3. Lemma.** *If  $(X, d)$  is a weakly convex compact metric space having more than one point, then  $r(X) < \delta(X)$ .*

*Proof.* By an induction argument one easily proves that for every  $n \geq 1$  and any  $x_1, \dots, x_n \in X$  there is a point  $z \in X$  such that for each  $w \in X$ ,

(CC1)  $d(z, w) \leq \max(d(x_1, w), \dots, d(x_n, w))$ ,

(CC2)  $d(x_1, w) = \dots = d(x_n, w)$  provided (CC1) is fulfilled with the equality sign.

Now suppose, for the contrary, that  $r(X) = \delta(X)$ . By the compactness, there is a maximal finite system  $x_1, \dots, x_n$  of elements of  $X$  such that  $d(x_j, x_k) = \delta(X)$  whenever  $j \neq k$ . Let  $z \in X$  be a point satisfying (CC1) and (CC2) for  $x_1, \dots, x_n$ . By our assumption,  $r_X(z) = \delta(X)$  and hence there is  $w \in X$  such that  $d(z, w) = \delta(X)$ . But then, by (CC2),  $d(x_1, w) = \dots = d(x_n, w) = \delta(X)$  which denies the maximality of the system  $x_1, \dots, x_n$ .  $\square$

**3.4. Proposition.** *For every nonempty weakly convex compact metric space  $X$  the set  $C^\infty(X)$  consists of a single point.*

*Proof.* By the compactness of  $X$  and the closedness of all  $C^n(X)$ 's,  $C^\infty(X)$  is nonempty and compact. It is easy to check that  $C^1(Y)$  is a fully convex subspace of  $Y$  for any weakly convex metric space  $Y$ . This yields that  $C^n(X)$  for each natural  $n$ , and thus  $C^\infty(X)$  as well, is a fully convex subspace of  $X$ . Take  $z \in C^\infty(X)$ . For every natural  $n$ ,  $z$  belongs to  $C(C^n(X))$  and hence there is  $y_n \in C^n(X)$  for which  $d(z, y_n) = r(C^n(X))$ . By the definitions of the Chebyshev center and the Chebyshev radius,  $r(Y) \geq \delta(C(Y))$  for every metric space  $Y$ . We infer from this that  $r(C^n(X)) \geq \delta(C^{n+1}(X))$  which implies that

$$(3-2) \quad d(z, y_n) \geq \delta(C^\infty(X)) \quad (n \in \mathbb{N}).$$

Now let  $(y_{n_k})_{k=1}^\infty$  be a subsequence of  $(y_n)_{n=1}^\infty$  which converges to some  $y \in X$ . Then  $y \in C^\infty(X)$  and hence  $d(z, y) = \delta(C^\infty(X))$ , thanks to (3-2). This shows that  $r_{C^\infty(X)}(z) = \delta(C^\infty(X))$  for every  $z \in C^\infty(X)$  and thus  $r(C^\infty(X)) = \delta(C^\infty(X))$ . Now it suffices to apply Lemma 3.3 to finish the proof.  $\square$

**3.5. Definition.** Let  $X$  be a nonempty weakly convex compact metric space. The unique point of  $C^\infty(X)$  is called the *generalized Chebyshev center* of  $X$  and is denoted by  $c_X$ .

The construction of the generalized Chebyshev center immediately gives

**3.6. Theorem.** *Let  $\mathcal{WCC}$  be the class of all nonempty weakly convex compact metric spaces. The assignment  $\mathcal{WCC} \ni X \mapsto c_X \in X$  is a strict distinguishing.*

The above result combined with Proposition 2.3 yields

**3.7. Corollary.** *Let  $X \in \mathcal{WCC}$ . For every isometry  $\Phi$  of  $X$  onto  $X$ ,  $\Phi(c_X) = c_X$ .*

When  $X$  is a convex subset of a normed linear space, Corollary 3.7 is a special case of Kakutani's fixed point theorem on equicontinuous group of affine transformations ([7]; or [17]). The proof presented here is constructive. However, it works only for the specific group — the isometry one.

**3.8. Remark.** The problem whether every (bijective) isometry between convex subsets of normed linear spaces is affine seems to be still open. (Beside the classical Mazur-Ulam theorem ([13]; or [1, 14.1]), the author knows only one general result [12] in this direction.) If there was a compact convex subset in a normed linear space admitting a non-affine isometry, then Corollary 3.7 would be stronger than Kakutani's fixed point theorem (in this specific case).

#### 4. CENTRAL MEASURE

In this section we apply the results of the previous part to distinguish a measure on a compact metric space. To do this, let us fix a nonempty compact metric space  $(X, d)$ . Denote by  $\text{Prob}(X)$  the set of all probabilistic Borel measures on  $X$ . Equip  $\text{Prob}(X)$  with the metric  $\widehat{d}$  given by the formula

$$(4-1) \quad \widehat{d}(\mu, \nu) = \sup \left\{ \left| \int_X f \, d\mu - \int_X f \, d\nu \right| : f \in \text{Contr}(X, \mathbb{R}) \right\}$$

where  $\mu, \nu \in \text{Prob}(X)$  and  $\text{Contr}(X, \mathbb{R})$  stands for the family of all  $d$ -nonexpansive maps of  $X$  into  $\mathbb{R}$ . The metric  $\widehat{d}$  is called the *Kantorovich* (or *Kantorovich-Rubinstein*, cf. [21, Definition 2.3.1]) metric induced by  $d$ . The space  $(\text{Prob}(X), \widehat{d})$  is compact and  $\widehat{d}$  induces on  $\text{Prob}(X)$  the topology inherited, thanks to the Riesz characterization theorem, from the weak-\* topology of the dual Banach space of  $\mathcal{C}(X, \mathbb{R})$ . It may be easily shown that  $(\text{Prob}(X), \widehat{d})$  is affinely isometric to a convex subset of a normed space. Therefore  $\text{Prob}(X)$  is weakly convex. We may now introduce

**4.1. Definition.** The generalized Chebyshev center of  $(\text{Prob}(X), \widehat{d})$  is called the *central measure* of  $X$  and it is denoted by  $\mu_X$ .

Now notice that every isometry  $\Phi: X \rightarrow X$  induces an affine isometry  $\widehat{\Phi}: \text{Prob}(X) \rightarrow \text{Prob}(X)$  given by the formula  $\widehat{\Phi}(\mu) = \mu \circ \Phi^{-1}$  where  $\mu \circ \Phi^{-1}$  denotes the transport of the measure  $\mu \in \text{Prob}(X)$  under the transformation  $\Phi$  (that is,  $(\mu \circ \Phi^{-1})(A) = \mu(\Phi^{-1}(A))$ ). We conclude from Corollary 3.7 that  $\widehat{\Phi}(\mu_X) = \mu_X$  for all  $\Phi \in \text{Iso}(X)$ ; that is,  $\mu_X$  is an invariant measure for the isometry group of  $X$ . Again, we have obtained a constructive proof that the isometry group of an arbitrary (nonempty) compact metric space admits an invariant measure.

Now suppose that  $\text{Iso}(X)$  acts transitively on  $X$ , i.e. for each two points  $x$  and  $y$  of  $X$  there is  $\Phi \in \text{Iso}(X)$  with  $\Phi(x) = y$ . It is known that in that case there is a unique measure invariant under every isometry of  $X$  (see e.g. [14, Theorem 2.5]). So, we get

**4.2. Proposition.** *If the isometry group of  $X$  acts transitively on  $X$ ,  $\mu_X$  is the unique measure invariant under every isometry of  $X$ .*

By a *metric group* we mean a metrizable topological group equipped with a left-invariant metric inducing the topology of the group (there exists one, see e.g. [2]). As a special case of Proposition 4.2 we obtain

**4.3. Corollary.** *Let  $G$  be a compact metric group. The central measure of  $G$  is the Haar measure of  $G$ .*

Corollary 4.3 provides a new constructive proof of the Haar measure theorem for metrizable compact groups.

Although in compact metric spaces with transitive actions of the isometry groups the central measures may be found thanks of their very specific properties, unfortunately computing a central measure in general is very complicated. For example, we do not know the one of  $[0, 1]$ . The reader interested in this problem may try first to compute the central measure of a three-point space.

**Question.** Is  $\mu_{[0,1]}$  the Lebesgue measure?

## 5. QUASI-NILPOTENT COMPACT METRIC SPACES

Although we do not know whether the central measure of the unit interval is the Lebesgue measure, we are able to make another distinguishing of measures in a special class of compact metric spaces in such a way that the distinguished measure for the unit interval will be the Lebesgue measure. This will be done in this section.

Recall (see Section 2) that for a metric space  $(X, d)$  the set  $X^{(1)}$  is the set of all orbits of points of  $X$  under the natural action of the isometry group of  $X$ . It turns out that  $X^{(1)}$  may be topologized by an ‘axiomatically’ defined metric when  $(X, d)$  is compact. Precisely, we denote by  $d^{(1)}$  the greatest pseudometric on  $X^{(1)}$  which makes the canonical projection  $\pi_X^{(1)}: (X, d) \rightarrow (X^{(1)}, d^{(1)})$  nonexpansive. For an

arbitrary metric space  $(X, d)$ ,  $d^{(1)}$  may not be a metric. However, we have

**5.1. Proposition.** *For every compact metric space  $(X, d)$ ,  $d^{(1)}$  is a metric on  $X^{(1)}$ . Moreover, for each  $x, y \in X$ ,*

$$(5-1) \quad d^{(1)}(\pi_X^{(1)}(x), \pi_X^{(1)}(y)) = \sup\{|f(\pi_X^{(1)}(x)) - f(\pi_X^{(1)}(y))| : \\ f: X^{(1)} \rightarrow \mathbb{R}, f \circ \pi_X^{(1)} \text{ is } d\text{-nonexpansive}\}.$$

*Proof.* The verification of (5-1) is left as a simple exercise. We shall only show that  $d^{(1)}$  is indeed a metric. We shall do this with use of the variation of the Gromov-Hausdorff metric [4, 5] (see also [16]). Namely, for  $a$  and  $b$  in  $X$  let  $\varrho(a, b)$  be the least upper bound of numbers  $p_H(X_1, X_2) + p((a, 1), (b, 2))$  where  $X_j = X \times \{j\}$ ,  $p$  is a semimetric on  $X_1 \cup X_2$  such that  $p((x, j), (y, j)) = d(x, y)$  for any  $x, y \in X$  and  $j \in \{1, 2\}$ , and  $p_H$  is the Hausdorff distance induced by  $p$ . (In other words,  $\varrho(a, b)$  is a counterpart of the Gromov-Hausdorff distance for pointed metric spaces  $(X, a)$  and  $(X, b)$ .) As in case of the classical Gromov-Hausdorff distance one shows that  $\varrho$  is a semimetric on  $X$  such that  $\varrho(a, b) = 0$  iff the pointed metric spaces  $(X, a)$  and  $(X, b)$  are isometric, i.e. if  $\Phi(a) = b$  for some  $\Phi \in \text{Iso}(X)$ . Thus  $\varrho$  induces a **metric**  $\varrho^*$  on  $X^{(1)}$  in such a way that  $\varrho^*(\pi_X^{(1)}(x), \pi_X^{(1)}(y)) = \varrho(x, y)$  for all  $x, y \in X$ . Since  $\varrho \leq d$  (because for  $p((x, 1), (y, 2)) := d(x, y)$  one obtains  $p_H(X_1, X_2) = 0$  and  $p(x, y) = d(x, y)$ ),  $\pi_X^{(1)}$  is nonexpansive with respect to the metrics  $d$  and  $\varrho^*$ , and thus  $d^{(1)} \geq \varrho^*$ .  $\square$

By Proposition 5.1,  $(X, d)^{(1)} := (X^{(1)}, d^{(1)})$  is a compact metric space provided  $(X, d)$  is so. Thus we may repeat this construction to obtain subsequent spaces  $X^{(2)}$ ,  $X^{(3)}$  and so on. Namely, for a compact metric space let  $(X^{(0)}, d^{(0)}) = (X, d)$  and  $(X^{(n)}, d^{(n)}) = (X^{(n-1)}, d^{(n-1)})^{(1)}$  for  $n > 0$ . Notice that  $\delta(X^{(n)}) \leq \delta(X^{(n-1)})$  and thus the sequence  $(\delta(X^{(n)}))_{n=1}^\infty$  is convergent. We introduce the following

**5.2. Definition.** A compact metric space  $X$  is said to be *quasi-nilpotent* iff  $\lim_{n \rightarrow \infty} \delta(X^{(n)}) = 0$ .

The class of quasi-nilpotent compact metric spaces includes all spaces on which their isometry groups acts transitively. One may think that such spaces have to have rich isometry groups. The next example shows that it is not the rule.

**5.3. Example.** Let  $(X, d)$  be the interval  $[a, b]$  with the natural metric. Observe that the isometry group of  $X$  is very poor — there is only one isometry on  $X$  different from the identity map. However,  $X$  is quasi-nilpotent. To see this, it suffices to show that  $X^{(1)}$  is isometric to  $[a/2, b/2]$ . But this may easily be shown by means of (5-1).

Now we shall distinguish a special measure on a quasi-nilpotent (non-empty) compact metric space, which may also be called central. For



a convex subset  $K$  of a normed linear space let  $\text{Fix}(K)$  be the set of all fixed points under every affine isometry of  $K$  onto  $K$ . The set  $K$  is convex as well (however, it may be empty). Further, let  $\text{Fix}^0(K) := K$  and for natural  $n > 0$  let  $\text{Fix}^n(K) = \text{Fix}(\text{Fix}^{n-1}(K))$ . Finally, put  $\text{Fix}^\infty(K) = \bigcap_{n=0}^\infty \text{Fix}^n(K)$ . Note that  $\text{Fix}^\infty(K)$  is convex and if  $K$  is compact,  $\text{Fix}^\infty(K)$  is nonempty.

Now let  $X$  be a nonempty compact metric space and  $\text{Prob}(X)$  be equipped with the Kantorovich metric induced by the metric of  $X$ . Let  $\Delta(X) = \text{Fix}^\infty(\text{Prob}(X))$ . In the sequel we shall prove that  $\Delta(X)$  consists of a single measure iff  $X$  is quasi-nilpotent. In fact, this follows from the following

**5.4. Theorem.** *For a nonempty compact metric space  $(X, d)$  the function*

$$\Psi: \text{Fix}(\text{Prob}(X)) \ni \mu \mapsto \mu \circ \pi_X^{-1} \in \text{Prob}(X^{(1)})$$

*is an affine isometry of  $\text{Fix}(\text{Prob}(X))$  onto  $\text{Prob}(X^{(1)})$ . In particular,  $\delta(\text{Fix}(\text{Prob}(X))) = \delta(X^{(1)})$ .*

*Proof.* Since  $\delta(\text{Prob}(Y)) = \delta(Y)$  for every compact metric space  $Y$ , it suffices to prove the first assertion. By [14],  $\Psi$  is an affine bijection. So, we only need to check that  $\Psi$  is isometric. Fix  $\mu_1, \mu_2 \in \text{Fix}(\text{Prob}(X))$  and put  $\nu_j = \Psi(\mu_j)$ . If  $u \in \text{Contr}(X^{(1)}, \mathbb{R})$ , then  $\int_{X^{(1)}} u d\nu_j = \int_X u \circ \pi_X^{(1)} d\mu_j$ . This, combined with (4-1), gives  $\widehat{d}(\mu_1, \mu_2) \geq \widehat{d}^{(1)}(\nu_1, \nu_2)$ . Conversely, if  $v \in \text{Contr}(X, \mathbb{R})$ , then, since  $\mu_j \in \text{Fix}(\text{Prob}(X))$ ,  $\int_X v d\mu_j = \int_X v \circ \Phi d\mu_j$  for every  $\Phi \in \text{Iso}(X)$ . Now by [15, Proposition 2.5], the closed convex hull (in the topology of uniform convergence) of the set  $\{v \circ \Phi: \Phi \in \text{Iso}(X)\}$  contains a map  $w: X \rightarrow \mathbb{R}$  such that

$$(5-2) \quad w \circ \Phi = w$$

for all  $\Phi \in \text{Iso}(X)$ . This implies that  $w \in \text{Contr}(X, \mathbb{R})$  and  $\int_X v d\mu_j = \int_X w d\mu_j$ . We infer from (5-2) that there is  $w_0: X^{(1)} \rightarrow \mathbb{R}$  such that  $w = w_0 \circ \pi_X^{(1)}$ . The latter connection and (5-1) yield that  $w_0 \in \text{Contr}(X^{(1)}, \mathbb{R})$ . So, we finally obtain  $|\int_X v d\mu_1 - \int_X v d\mu_2| = |\int_X w_0 \circ \pi_X^{(1)} d\mu_1 - \int_X w_0 \circ \pi_X^{(1)} d\mu_2| = |\int_{X^{(1)}} w_0 d\nu_1 - \int_{X^{(1)}} w_0 d\nu_2| \leq \widehat{d}^{(1)}(\nu_1, \nu_2)$ , which finishes the proof.  $\square$

Now Theorem 5.4 and induction argument give

**5.5. Proposition.** *If  $X$  is a nonempty compact metric space, then  $\delta(\text{Fix}^n(\text{Prob}(X))) = \delta(X^{(n)})$  for each natural  $n$ .*

**5.6. Corollary.** *Let  $X$  be a nonempty compact metric space.  $\Delta(X)$  consists of a single measure iff  $X$  is quasi-nilpotent.*

**5.7. Definition.** Let  $X$  be a nonempty quasi-nilpotent compact metric space. The unique member of  $\Delta(X)$  is denoted by  $\lambda_X$  and it is called the *central measure of  $X$  of a second kind*.

Since  $\lambda_X \in \text{Fix}(\text{Prob}(X))$ , the central measure of  $X$  of a second kind is invariant under every isometry of  $X$ . We conclude from this that  $\lambda_X = \mu_X$  provided  $X$  is a compact metric space such that  $X^{(1)}$  is a singleton (i.e. if the isometry group of  $X$  acts transitively on  $X$ ). We end the section with

**5.8. Proposition.** *The central measure of  $[0, 1]$  of a second kind coincides with the Lebesgue measure on  $[0, 1]$ .*

*Proof.* For simplicity, put  $I = [0, 1]$  and  $\lambda = \lambda_I$ . We infer from the relation  $\lambda \in \text{Fix}(\text{Prob}(I))$  that  $\lambda = \lambda \circ u_1$  where  $u_1: I \ni t \mapsto |t - 1/2| \in [0, 1/2]$ . Similarly, since  $\lambda \in \text{Fix}^n(\text{Prob}(I))$ ,  $\lambda$  is invariant under the map  $u_n: [0, 1/2^{n-1}] \ni t \mapsto |t - 1/2^n| \in [0, 1/2^n]$ . One deduces from this that  $\lambda(\{\frac{k}{2^n}\}) = 0$  and  $\lambda([\frac{k-1}{2^n}, \frac{k}{2^n}]) = \frac{1}{2^n}$  for each natural  $k$  and  $n$  with  $1 \leq k \leq 2^n$ , and hence  $\lambda$  is the Lebesgue measure. The details are left for the reader.  $\square$

## 6. DISTINGUISHING DENSE SUBSETS

We know that strict distinguishing is impossible in general. However, one may still ask whether it is possible to define a distinguishing in the class  $\mathcal{K}$  of all nonempty compact metric spaces. This part is devoted to the solution of this problem. We shall show that there is a sequence of distinguishings  $\mathcal{K} \ni K \mapsto C_n(K) \in K^{(1)}$  ( $n \geq 1$ ) such that for every  $K \in \mathcal{K}$ , whenever  $c_n \in K$  satisfies  $\pi_K^{(1)}(c_n) = C_n(K)$ , then the set  $\{c_n: n \geq 1\}$  is dense in  $K$ .

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Fix an infinite compact metric space  $(X, d)$ . Instead of constructing an ‘intrinsic’ dense subset of  $X$ , we shall construct a metric  $\varrho_X$  on  $\mathbb{N}$  such that  $(\mathbb{N}, \varrho_X)$  is isometric to a dense subset of  $X$ . Suppose for some  $n \in \mathbb{N}$  we have defined the metric  $\varrho_X$  on  $\{0, \dots, n\}$  in such a way that the space  $(\{0, \dots, n\}, \varrho_X)$  is isometrically embeddable into  $X$  (for  $n = 0$  we have nothing to do). Put

(6-1)

$$F_n := \{(x_0, \dots, x_n) \in X^{n+1}: d(x_j, x_k) = \varrho_X(j, k), j, k = 0, \dots, n\}$$

and

$$(6-2) \quad f_n: F_n \times X \ni (x_0, \dots, x_n; x) \mapsto \min(d(x_0, x), \dots, d(x_n, x)) \in \mathbb{R}.$$

By our assumption,  $F_n$  is nonempty. Next, let

$$(6-3) \quad A_0^{n+1} := \{(x; y) \in F_n \times X: f_n(x; y) = \max f_n(F_n \times X)\}.$$

Now inductively define sets  $A_j^{n+1}$  for  $j = 1, \dots, n+1$  by

$$(6-4) \quad A_j^{n+1} := \left\{ (y_0, \dots, y_n; y) \in A_{j-1}^{n+1}: \right. \\ \left. d(y_{j-1}, y) = \max\{d(x_{j-1}, x) \mid (x_0, \dots, x_n; x) \in A_{j-1}^{n+1}\} \right\}.$$

The reader will easily check (by induction and the compactness argument) that each of the sets  $A_j^{n+1}$ 's is nonempty. Now take an arbitrary  $(x_0, \dots, x_n; x_{n+1}) \in A_{n+1}^{n+1}$  and put  $\varrho_X(j, n+1) := d(x_j, x_{n+1})$  for  $j = 0, \dots, n+1$ . Observe that this definition is independent of the choice of  $(x_0, \dots, x_n; x) \in A_{n+1}^{n+1}$ . It is also clear that  $\varrho_X$  is a metric (not only a semimetric) on  $\{0, 1, \dots, n+1\}$  (because  $X$  is infinite).

In this way we obtain a metric  $\varrho_X$  on  $\mathbb{N}$  such that  $\varrho_X = \varrho_Y$  for every space  $Y$  isometric to  $X$ . We claim that

**6.1. Proposition.** *For every infinite compact metric space  $(X, d)$ , the space  $(\mathbb{N}, \varrho_X)$  is isometric to a dense subset of  $X$ .*

*Proof.* For each  $n \in \mathbb{N}$  let  $P_n$  be the set of all sequences  $(x_m)_{m=0}^\infty \in X^\mathbb{N}$  such that the function  $(\{0, \dots, n\}, \varrho_X) \ni j \mapsto x_j \in (X, d)$  is isometric. By construction of  $\varrho_X$ ,  $P_n$  is nonempty. It is also clear that  $P_n$  is closed in  $X^\mathbb{N}$  and that  $P_n \supset P_{n+1}$ . Therefore, by the compactness of  $X$ , the intersection  $\bigcap_{n=0}^\infty P_n$  is nonempty. We infer from this that there is an isometric function  $\Phi$  of  $(\mathbb{N}, \varrho_X)$  into  $(X, d)$ . We claim that  $\Phi(\mathbb{N})$  is dense in  $X$ . Suppose, for the contrary, that there is  $x \in X$  and  $r > 0$  such that

$$(6-5) \quad d(x, \Phi(n)) \geq r$$

for every  $n \in \mathbb{N}$ . Note that  $(\Phi(0), \dots, \Phi(n+1)) \in A_{n+1}^{n+1} \subset A_0^{n+1} \subset F_n \times X$  for any  $n \in \mathbb{N}$ , where  $A_j^{n+1}$ 's and  $F_n$ 's are given by (6-4), (6-3) and (6-1). So, (6-5) yields  $\max f_n(F_n \times X) \geq r$  for  $f_n$ 's given by (6-2). Finally, we conclude from the relation  $(\Phi(0), \dots, \Phi(n+1)) \in A_0^{n+1}$  and (6-5) that  $f_n(\Phi(0), \dots, \Phi(n); \Phi(n+1)) \geq r$  which means that  $d(\Phi(j), \Phi(k)) \geq r$  for  $j < k$ . But this denies the compactness of  $X$ .  $\square$

By a *representation* of the metric  $\varrho_X$  we mean any isometric function of  $(\mathbb{N}, \varrho_X)$  into  $(X, d)$ , provided  $X$  is infinite.

If  $X$  is finite and has  $n$  elements, we may repeat the above construction to obtain a metric  $\varrho_X$  on  $\{0, \dots, n-1\}$  which makes this set isometric to  $X$ . In that case by a *representation* of  $\varrho_X$  we mean any function  $\Phi: \mathbb{N} \rightarrow X$  such that  $\Phi$  is isometric on  $\{0, \dots, n-1\}$  (with respect to the metrics  $\varrho_X$  and  $d$ ) and  $\Phi(k) = \Phi(n-1)$  for  $k > n-1$ .

We may ask how many representations has the metric  $\varrho_X$  for an arbitrary space  $X$ . The answer to this gives

**6.2. Proposition.** *Let  $X$  be a nonempty compact metric space and  $\Phi_0: \mathbb{N} \rightarrow X$  a representation of  $\varrho_X$ . The function  $\Psi \mapsto \Psi \circ \Phi_0$  establishes a one-to-one correspondence between isometries  $[\Psi]$  of  $X$  and representations  $[\Psi \circ \Phi_0]$  of  $\varrho_X$ .*

The above result is an immediate consequence of Proposition 6.1 (and the well known fact that every isometric map of a compact metric space into itself is onto [11]). Proposition 6.2 says that if the isometry group of a space  $X$  is poor, there are only few representations of  $\varrho_X$ .

In the opposite, if there are many representations, the isometry group of  $X$  is rich. Both the situations are interesting.

Now we pass to distinguishing of points. Observe that whatever representation  $\Phi: \mathbb{N} \rightarrow X$  of  $\varrho_X$  we take, the function  $\pi_X^{(1)} \circ \Phi$  is the same. It follows from the latter that the definition  $C_n(X) := \pi_X^{(1)}(\Phi(n))$  where  $n \in \mathbb{N}$  and  $\Phi$  is any representation of  $\varrho_X$  is correct. We now clearly have

**6.3. Proposition.** *For each  $n \in \mathbb{N}$ , the assignment  $\mathcal{K} \ni K \mapsto C_n(K) \in K^{(1)}$  is a distinguishing.*

In studying the class of separable complete metric spaces, especially in theory of random metric spaces (cf. [19, 20]), one of methods is to consider the set of all metrics  $\mathfrak{D}$  on  $\mathbb{N}$  and to make the assignment  $\mathfrak{D} \ni d \mapsto$  ‘the completion of  $(\mathbb{N}, d)$ ’. In other words, the ‘world’ of infinite separable complete metric spaces may be identified (by this assignment) with the ‘world’ of metrics on  $\mathbb{N}$ . This is quite natural approach, however, there is no one-to-one correspondence between the members of these two worlds. The distinguishing of dense subsets of compact metric spaces constructed in this section may be seen as an example of the ‘inverse function’ to the above assignment after restricting the considerations to totally bounded metrics on  $\mathbb{N}$ .

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